§4. Effective BV Quantization

Last time: Homotopy Lie algebra $\mathfrak{g}$

$$\Updownarrow$$

Vector field δ on $\mathfrak{g}[1]$ s.t. $\delta^2 = 0$

We can write $\delta = \delta_1 + \delta_2 + \cdots$ where

$$\delta_k : \mathfrak{g}^\vee[-1] \longmapsto \mathrm{Sym}^k(\mathfrak{g}^\vee[-1])$$

- Lie algebra: $\delta = \delta_2$
- DGLA: $\delta = \delta_1 + \delta_2$

Convention: Given $A, B \in \mathrm{Hom}(V, V)$, we write the commutator $[A, B] := AB - (-1)^{|A||B|} BA$

where $|A|$ is the degree of A. In particular,

$[A, B] = AB + BA$ if A, B are odd operators

• ## BV master equation

<u>Def'n</u>: A DGBV algebra is a triple (A, Q, Δ) where

① A: graded commutative associative algebra

② $Q: A \longmapsto A$ derivation of $\deg = 1$, $Q^2 = 0$

③ $\Delta: A \longmapsto A$ "2nd order" operator of $\deg = 1$, $\Delta^2 = 0$.
(BV operator)

④ $[Q, \Delta] = Q\Delta + \Delta Q = 0$.

Here Δ being "2nd order" means the following:

Define the "BV bracket"

$$\{a, b\} := \Delta(ab) - (\Delta a) b - (-1)^{|a|} a \Delta b \quad \forall a, b \in A$$

(the failure of Δ being a derivation)

then $\{-,-\}: A \otimes A \longmapsto A$ is $\deg = 1$ satisfying

1) $\{a, b\} = (-1)^{|a||b|} \{b, a\}$

2) $\{a, bc\} = \{a, b\} c + (-1)^{(|a|+1)|b|} b \{a, c\}$

3) $\Delta\{a, b\} = -\{\Delta a, b\} - (-1)^{|a|} \{a, \Delta b\}$

Eg. X smooth manifold. Ω volume form

$$\Rightarrow \quad (PV^{\cdot}(X), \Delta_\Omega) \qquad \Delta_\Omega : PV^k \to PV^{k-1} \text{ divergence operator}$$

Define $\{\alpha, \beta\} := \Delta_\Omega(\alpha\beta) - (\Delta_\Omega \alpha)\beta - (-1)^{|\alpha|} \alpha \Delta_\Omega \beta$

Then $\{-,-\}$ = Schouten-Nijenhuis bracket (up to a sign)

Fact: $\{-,-\}$ here doesn't depend on the choice of Ω.

Eg: X Calabi-Yau manifold. Ω holomorphic volume form

$$PV^{k,l}(X) = \Omega^{0,l}(X, \wedge^k T_X^{1,0})$$

$\bar{\partial} : PV^{k,l} \mapsto PV^{k,l+1}$ Dolbeault differential

$\Delta : PV^{k,l} \mapsto PV^{k-1,l}$ divergence operator w.r.t. Ω

$$\Rightarrow \quad (PV^{-\bullet,\bullet}(X), \bar{\partial}, \Delta) \quad \text{DGBV}$$

RK: The existence of such structure implies that the local moduli of complex str. of C-Y X is smooth (BTT Lemma)

<u>Def'n</u>: Let (A, Q, Δ) be a DGBV. Let

$$I_0 \in A_0 \quad (\text{so } \deg(I_0) = 0)$$

I_0 is said to satisfying classical master equation (CME) if

$$Q I_0 + \frac{1}{2}\{I_0, I_0\} = 0$$

This implies $\left(Q + \{I_0, -\}\right)^2 = 0$ on $\mathcal{A}$

In good situation,

$$Q + \{\underbrace{I_0}_{\text{interaction part}}, -\} = \{\underbrace{S_0}_{\text{classical action}}, -\}$$

then CME: $\{S_0, S_0\} = 0$

Upshot: Any classical action / gauge symmetry

$\Rightarrow$ a solution of CME

Def'n: $I \in A_0[[\hbar]]$ is said to satisfy

quantum master equation (QME) if

$$QI + \hbar \Delta I + \frac{1}{2}\{I, I\} = 0$$

$$\left(I = I_0 + \hbar I_1 + \hbar^2 I_2 + \cdots \right)$$

$$\Big\downarrow \; \hbar \mapsto 0$$

$$Q I_0 + \frac{1}{2}\{I_0, I_0\} = 0$$

In good situation, $Q + \{I, -\} = \{S, -\}$

QME : $\hbar \Delta S + \frac{1}{2}\{S, S\} = 0$

In QFT, quantization process asks

$$S_0 \rightsquigarrow S = S_0 + \hbar S_1 + \cdots$$

classical action $\qquad\qquad$ quantum action

CME $\qquad\qquad$ QME

$\underline{\underline{\text{Ex}}}$: QME $\iff$ $(Q+\hbar\Delta)e^{I/\hbar} = 0$

(or in good situation $\Delta e^{S/\hbar} = 0$)

· (-1)-shifted symplectic geometry (Toy model)

Let (V, Q, ω) be a finite dim'l dg symplectic space

- $Q : V \mapsto V$ differential
- $\omega : \Lambda^2 V \mapsto \mathbb{R}/\mathbb{C}$ non-degenerate pairing of $\deg = -1$

$\left(\omega(a,b) = 0 \text{ unless } |a| + |b| = 1 \right)$

- $Q(\omega) = 0$ Explicitly

$$\omega(Q(a), b) + (-1)^a \omega(a, Q(b)) = 0$$

Non-degeneracy of ω

$$\Rightarrow \omega : \underset{(\text{dual})}{V^\vee} \xrightarrow{\sim} V[1] \quad \text{isom.}$$

This allows us to identify

$$\Lambda^2 V^\vee \xleftrightarrow{\sim} \Lambda^2(V[1]) \simeq \mathrm{Sym}^2(V)[2]$$

$$\omega \longleftrightarrow K[2]$$

$K = \omega^{-1} \in \mathrm{Sym}^2(V)$ is the Poisson kernel

$$\deg(K) = 1 \qquad Q(K) = 0$$

We obtain a DGBV $(\mathcal{A}, Q, \Delta_K)$ as follows

- $\mathcal{A} = \mathcal{O}(V) = \widehat{\mathrm{Sym}}(V^\vee)$ formal functions on V
- $Q : \mathcal{A} \mapsto \mathcal{A}$ induced from $Q : V \to V$
- $\Delta_K : \mathrm{Sym}^m(V^\vee) \longmapsto \mathrm{Sym}^{m-2}(V^\vee)$ is the contraction w/. the Poisson kernel $K \in \mathrm{Sym}^2(V)$

Explicitly, for $\alpha_i \in V^\vee$

$$\Delta_K(\alpha_1 \otimes \cdots \otimes \alpha_m)$$

$$= \sum_{i<j} \pm \langle K, \alpha_i \otimes \alpha_j \rangle \alpha_1 \otimes \cdots \otimes \hat{\alpha}_i \otimes \cdots \otimes \hat{\alpha}_j \otimes \cdots \otimes \alpha_m$$

Here $\pm$ is the Koszul sign by permuting graded objects.

Prop: $(\mathcal{O}(V), Q, \Delta_K)$ is a DGBV

Pf: Exercise. #

Let $S_0 \in \mathcal{O}(V)$ solve the CME:

$$\{S_0, S_0\} = 0. \quad (\deg S_0 = 0)$$

Then $\delta = \{S_0, -\}$ defines a vector field of V w/.

$$\deg \delta = 1 \quad , \delta^2 = 0$$

$$\Rightarrow (\mathfrak{g} = V[-1], \delta) \text{ is an } L_\infty\text{-algebra.}$$

· Field theory (BV formalism)

A classical field theory can be usually organized into an ∞-dim'l (-1)-symplectic geometry

$$(\mathcal{E}, Q, \omega) \quad \text{where}$$

① $\mathcal{E} = \Gamma(X, E^{\bullet})$ fields. $E^{\bullet}$ graded vector bundles

② $(\mathcal{E}, Q)$ elliptic complex

$$\longrightarrow \mathcal{E}^{-1} \xrightarrow{Q} \mathcal{E}^{0} \xrightarrow{Q} \mathcal{E}^{1} \rightarrow \cdots \qquad \left(\begin{array}{l} \text{Eg. } Q = \partial \\ \phantom{\text{Eg. }} Q = \bar{\partial} \end{array} \right)$$

③ ω: local (-1)-symplectic pairing

$$\omega(\alpha, \beta) = \int_X \langle \alpha, \beta \rangle \quad \forall \alpha, \beta \in \mathcal{E}$$

compatible w/ Q

$$\omega(Q(\alpha), \beta) + (-1)^{\alpha} \omega(\alpha, Q(\beta)) = 0$$

To describe the quantization, we perform the Toy model

- $\mathcal{E}^\vee := \mathrm{Hom}_X(\mathcal{E}, \mathbb{R})$ distribution

- $(\mathcal{E}^\vee)^{\otimes n} = \mathrm{Hom}_{X^n}(\mathcal{E}^{\otimes n}, \mathbb{R})$ distribution on X^n

$$\Longrightarrow \quad \mathrm{Sym}^n(\mathcal{E}^\vee) \text{ is defined.}$$

Let us form

$$\mathcal{O}(\mathcal{E}) := \prod_{n \geq 0} \mathrm{Sym}^n(\mathcal{E}^\vee)$$

Let $\mathcal{O}_{loc}(\mathcal{E}) \subset \mathcal{O}(\mathcal{E})$ be local functionals

$$\left\{ \int_X \mathcal{L} \;\middle|\; \mathcal{L} : \text{Lagrangian density} \right\}$$

- $K = \omega^{-1}$ δ-function distribution

$$\left(f(x) = \int dy\, f(y)\, \delta_{x,y} \right)$$

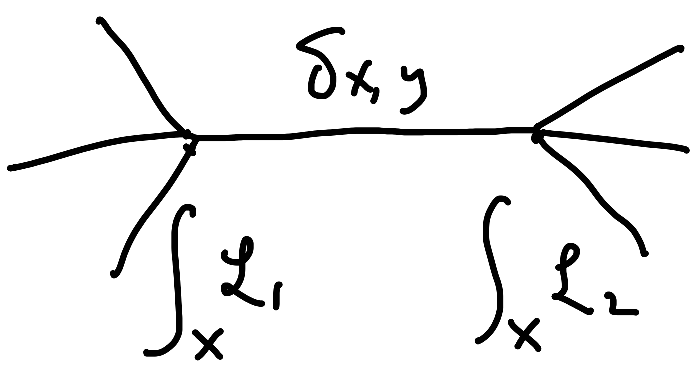
K is a distributional section of $\mathrm{Sym}^2(\mathcal{E})$

Problem: $\Delta_k \curvearrowright \mathcal{O}(\mathcal{E})$ ill-defined

Ultra-Violet Problem

Ex: The corresponding BV bracket $\{-,-\}$ is well-defined on local functionals $\mathcal{O}_{loc}(\mathcal{E})$

$$\{-,-\}: \mathcal{O}_{loc}(\mathcal{E}) \times \mathcal{O}_{loc}(\mathcal{E}) \longmapsto \mathcal{O}_{loc}(\mathcal{E})$$



$$= \int_x (\cdots)$$

$\Rightarrow$ CME makes sense for local functionals.

But for quantization: $I_0 \to I = I_0 + \hbar I_1 + \cdots$

$$\text{“} QI + \hbar \Delta I + \frac{1}{2}\{I,I\} = 0 \text{”}$$

problematic

Solution: renormalization

Eg: [CS Theory] X 3-mfd. $\mathfrak{g}$: Lie algebra

Tr: Killing pairing

- $\mathcal{E} = \Omega^\bullet(X, \mathfrak{g}[1])$

	$\Omega^0(X,\mathfrak{g})$	$\Omega^1(X,\mathfrak{g})$	$\Omega^2(X,\mathfrak{g})$	$\Omega^3(X,\mathfrak{g})$
deg:	-1	0	1	2
	ghost	field (connection)	anti-field	anti-ghost

- $\omega(\alpha, \beta) = \pm \int_X \mathrm{Tr}\langle \alpha, \beta \rangle$ $\quad \deg \omega = -1$

- $$CS[\mathcal{A}] = \int \mathrm{Tr}\Big(\underbrace{\tfrac{1}{2}\mathcal{A} \wedge d\mathcal{A}}_{\text{(free part)}} + \underbrace{\tfrac{1}{6}\mathcal{A} \wedge [\mathcal{A}, \mathcal{A}]}_{\text{(interaction)}} \Big)$$

$$\mathcal{A} = \underset{\Omega^0}{c} + \underset{\Omega^1}{A} + \underset{\Omega^2}{A^\vee} + \underset{\Omega^3}{c^\vee} \in \mathcal{E}$$

$$= \int \mathrm{Tr}\left(\tfrac{1}{2} A \wedge dA + \tfrac{1}{6} A \wedge [A, A]\right) + \text{terms containing ghosts}$$

Claim: CS satisfies CME

$$\{CS, CS\} = 0$$

If we write $CS = \text{free} + I$ as above.

then $\{\text{free}, -\} = d$

$$CME \iff dI + \frac{1}{2}\{I, I\} = 0$$

One way to see this is that

$$\Omega^\bullet(X, \mathfrak{g}) \text{ is a DGLA} \implies CME.$$

- Costello's Homotopic renormalization

we have seen that

$$\Delta_K \circlearrowright \mathcal{O}(\mathcal{E}) \quad \text{ill-defined naively}$$

$$K = \omega^{-1} \quad \text{singular}$$

Toy model: (V, Q) finite dim'l

$K_0 \in \mathrm{Sym}^2(V)$ "Poisson kernel"

$\deg K_0 = 1 \qquad Q(K_0) = 0$

$\Rightarrow \quad \Delta_0 : \mathcal{O}(V) \to \mathcal{O}(V)$ BV operator

by contracting w/. K_0

Let $P \in \mathrm{Sym}^2(V)$ $\deg P = 0$. Consider the

chain homotopy for BV kernel

$$K_P = K_0 + Q(P) = K_0 + (Q \otimes 1 + 1 \otimes Q) P$$

then $\deg(K_P) = 1 \qquad Q(K_P) = QK_0 + Q^2(P) = 0$

Consider new BV operator

$$\Delta_P = \text{contraction w/. } K_P$$

Prop: $$(Q+\hbar\Delta_P)\, e^{\hbar\partial_P} = e^{\hbar\partial_P}\,(Q+\hbar\Delta_0)$$

Here ∂_P is the 2nd order operator of contracting w/. $P \in \mathrm{Sym}^2(V)$

$$\partial_P : \mathrm{Sym}^n(V^\vee) \mapsto \mathrm{Sym}^{n-2}(V^\vee)$$

$$\begin{tikzcd}
\mathcal{O}(V)[[\hbar]] \arrow[r, "e^{\hbar\partial_P}"] \arrow[d, "Q+\hbar\Delta_0"'] & \mathcal{O}(V)[[\hbar]] \arrow[d, "Q+\hbar\Delta_P"] \\
\mathcal{O}(V)[[\hbar]] \arrow[r, "e^{\hbar\partial_P}"] & \mathcal{O}(V)[[\hbar]]
\end{tikzcd}$$

this implies that

$$\begin{tikzcd}
\text{QME} \arrow[r, "e^{\hbar\partial_P}"] & \text{QME} \\[-20pt]
(\mathcal{O}(V), Q, \Delta_0) & (\mathcal{O}(V), Q, \Delta_P)
\end{tikzcd}$$

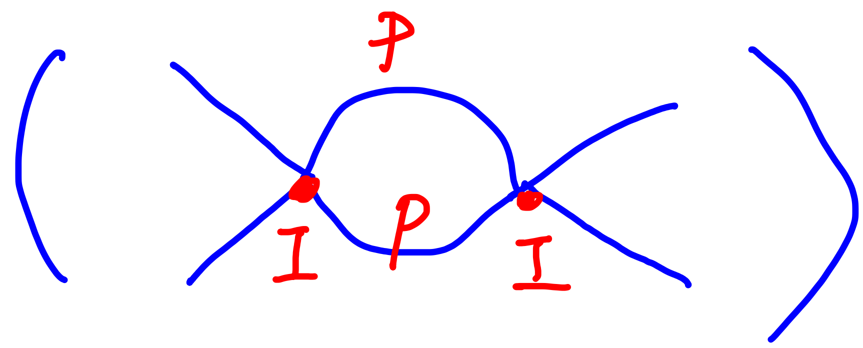
Homotopy RG flow

$$(Q + \hbar \Delta_0) e^{I/\hbar} = 0 \iff (Q + \hbar \Delta_P) e^{\tilde{I}/\hbar} = 0$$

where $e^{\tilde{I}/\hbar} = e^{\hbar \partial_P} e^{I/\hbar}$.

As we have seen before, this can be expressed by

$$\tilde{I} = \sum_{\text{conn graph}}$$



Feynman Diagrams

- Back to QFT $(\mathcal{E} = \Gamma(X, E^\bullet), Q, \omega)$

$K_0 = \omega^{-1}$ δ-function distribution

$Q(K_0) = 0$

Costello's approach: using

$$H^\bullet(\text{distribution}, Q) = H^\bullet(\text{Smooth}, Q)$$

(elliptic regularity)

$$K_0 = \underset{\text{smooth}}{K_r} + Q\,\underset{\text{singular (parametrix)}}{P_r}$$

$\Delta_r : \mathcal{O}(\mathcal{E}) \mapsto \mathcal{O}(\mathcal{E})$ contracting w/. K_r

is well-defined since K_r is smooth,

$\Rightarrow (\mathcal{O}(\mathcal{E}), Q, \Delta_r)$ "effective" DGBV

Let r' be another regularization

$$K_0 = K_{r'} + Q P_{r'}$$

$$\Rightarrow K_{r'} - K_r = Q \underset{\text{smooth}}{P_r^{r'}}$$

Let $\partial_{P_r^{r'}} : \mathcal{O}(\mathcal{E}) \to \mathcal{O}(\mathcal{E})$ be contracting

w/. smooth kernel $P_r^{r'}$

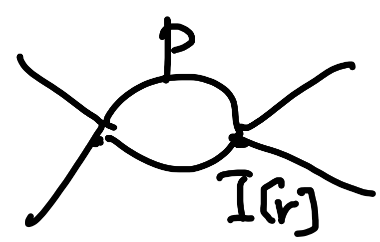
$$\Rightarrow (\mathcal{O}(\mathcal{E})[[\hbar]], Q + \hbar\Delta_r) \xrightarrow{e^{\hbar \partial_{P_r^{r'}}}} (\mathcal{O}(\mathcal{E})[[\hbar]], Q + \hbar\Delta_{r'})$$

(Homotopy RG flow)

<u>Def'n</u> [Costello] An effective sol'n of perturbative BV quantization of I_0 (which solves CME) is given by a family $I[r] \in \mathcal{O}(\mathcal{E})[[\hbar]]$ (for each choice of regularization P_r) such that

① $(Q + \hbar \Delta_r) e^{I[r]/\hbar} = 0$ (effective QME)

② $e^{I[r']/\hbar} = e^{\hbar \partial_{P_r^{r'}}} e^{I[r]/\hbar}$ (Homotopy RG)

equivalently $I[r'] = \sum_{\text{conn graph}} \Big($ 

$\Big)$

③ $I[r]$ is asymptotic local when $r \to 0$ and

$$I_0 = \lim_{r \to 0} \lim_{\hbar \to 0} I[r].$$

<u>Ref Today</u> :

Costello : Renormalization and effective field theory
(for Costello's homotopy renormalization theory)

S. Li : vertex algebras and quantum master equation
(for the style of the current presentation)